TOPOLOGY - III, EXERCISE SHEET 5

Exercise 1. *Warm up* (easy)

(1) Let X be a contractible space, show that $H_n(X) = 0$ for all $n > 0$.
(2) Show that for every space X, the cone $CX := (X \times [0,1])/(X \times \{0\})$ is contractible.
(3) Show that the relative homology group $H_0(X, A) = 0$ for a space X and a sub-space $A \subseteq X$ if and only if A meets X at every path connected component of X.

Exercise 2. *Homomorphism on homology induced by inclusion map.* (easy)

(1) Recall that a subspace $A \subseteq X$ is called a retract of X if there exists a continuous map $r : X \to A$ such that $r \circ i = Id_A$, where $i : A \hookrightarrow X$ is the inclusion map. Show that if a subspace A is a retract of a space X, then the induced map $i_* : H_n(A) \to H_n(X)$ is injective for all n.
(2) Give an example of a subspace A of a space X such that the map $i_* : H_n(A) \to H_n(X)$ is not injective for some n.

Exercise 3. *Computation of some relative homology groups* (medium)
Compute the relative homology groups $H_i(X, A)$ for the following pairs of spaces with $A \subseteq X$:

(1) (S^2, A) where A is a finite subset of points of S^2.
(2) $(D^2, \partial D^2)$, where D^2 is the closed unit disc in $\mathbb{R}^2$.
(3) $(\mathbb{T}^2, A)$, where A is a finite set of points.
(4) $(\mathbb{T}^2, S^1)$ where the copy of $S^1 \subseteq \mathbb{T} := S^1 \times S^1$ is given by the diagonal inclusion:
$$i : S^1 \to S^1 \times S^1,\ z \mapsto (z, z).$$
What are the relative homology groups if the inclusion is given by $i(z) = (z, p)$ for a fixed $p \in S^1$?
(5) $(\mathbb{R}, \mathbb{Q})$.

Exercise 4. (medium)
Through this exercise we will fill in the details of the proof of the following fact:
Let $F : X \times I \to Y$ be a homotopy between the maps $f, g : X \to Y$. Then f, g induce homotopy equivalent morphisms of complexes $C_\bullet(X) \to C_\bullet(Y)$.

(1) Consider the prism $\Delta^n \times [0,1]$. We label $\Delta^n \times \{0\}$ as $[v_0, ..., v_n]$ and label $\Delta^n \times \{1\}$ as $[w_0, ..., w_n]$ with w_i lying over v_i. For $i = 0, ..., n$, let Γ_i be the $(n+1)$-simplex $[v_0, ..., v_i, w_i, ..., w_n]$. Show that Γ_i and Γ_j intersect at an $n-$simplex if and only if

$|i - j| = 1$. Also prove that

$$\Delta^n \times [0,1] = \bigcup_{i=0}^{n} \Gamma_i.$$

Hint: Draw a picture for $n = 1, 2$.

(2) For all n, consider the map

$$P : C_n(X) \to C_{n+1}(Y),\ \sigma \mapsto \sum_{i=0}^{n} (-1)^n F(\sigma \times Id_{[0,1]})_{|\Gamma_i}.$$

Show that P is a homotopy between the complex maps g_* and f_*. That is:

$$g_* - f_* = \partial P + P\partial.$$